

STEIN-WEISS AND CAFFARELLI-KOHN-NIRENBERG INEQUALITIES WITH ANGULAR INTEGRABILITY

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ABSTRACT. We prove an extension of the Stein-Weiss weighted estimates for fractional integrals, in the context of L^p spaces with different integrability properties in the radial and the angular direction. In this way, the classical estimates can be unified with their improved radial versions. A number of consequences are obtained: in particular we deduce precised versions of weighted Sobolev embeddings, Caffarelli-Kohn-Nirenberg estimates, and Strichartz estimates for the wave equation, which extend the radial improvements to the case of arbitrary functions.

1. INTRODUCTION

The radial estimate of Walter Strauss [19] states that for radial functions $u \in \dot{H}^1(\mathbb{R}^n)$, $n \geq 2$, one has

$$|x|^{\frac{n-1}{2}}|u(x)| \leq C\|\nabla u\|_{L^2}, \quad |x| \geq 1. \quad (1.1)$$

This is an example of a well known general phenomenon: under suitable assumptions of symmetry, notably radial symmetry, classical estimates and embeddings of spaces admit substantial improvements. In the case of (1.1), a control on the H^1 norm of u gives a pointwise bound and decay of u , which are false in the general case. Radial and more general symmetric estimates have been extensively investigated, in view of their relevance for applications, especially to differential equations.

This phenomenon is quite natural; indeed, symmetric functions can be regarded as functions defined on lower dimensional manifolds, hence satisfying stronger estimates, extended by the action of some group of symmetries. Radial functions are essentially functions on \mathbb{R}^+ , while the norms on \mathbb{R}^n introduce suitable dimensional weights connected to the volume form.

In view of the gap between the symmetric and the non symmetric case, an interesting question arises: is it possible to quantify the defect of symmetry of functions and prove more general estimates which encompass all cases, and in particular reduce to radial estimates when applied to radial functions? Heuristically, one should be able to improve on the general case by introducing some measure of the distance from the maximizers of the inequality, which typically have the greatest symmetry.

The aim of this paper is to give a partial positive answer to this question, through the use of the following type of mixed radial-angular norms:

$$\|f\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}} = \left(\int_0^{+\infty} \|f(\rho \cdot)\|_{L^{\tilde{p}}(\mathbb{S}^{n-1})}^p \rho^{n-1} d\rho \right)^{\frac{1}{p}}, \quad \|f\|_{L_{|x|}^{\infty} L_{\theta}^{\tilde{p}}} = \sup_{\rho > 0} \|f(\rho \cdot)\|_{L^{\tilde{p}}(\mathbb{S}^{n-1})}.$$

When the context is clear we shall write simply $L^p L^{\tilde{p}}$. For $p = \tilde{p}$ the norms reduce to the usual L^p norms

$$\|u\|_{L_{|x|}^p L_{\theta}^p} \equiv \|u\|_{L^p(\mathbb{R}^n)},$$

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while for radial functions the value of \tilde{p} is irrelevant:

$$u \text{ radial} \implies \|u\|_{L^p L^{\tilde{p}}} \simeq \|u\|_{L^p(\mathbb{R}^n)} \quad \forall p, \tilde{p} \in [1, \infty].$$

Notice also that the norms are increasing in \tilde{p} . The idea of distinguishing radial and angular directions is not new and has proved successful in the context of Strichartz estimates and dispersive equations (see [12], [18], [3]; see also [2]). To give a flavour of the results which can be obtained, Strauss' estimate (1.1) can be extended as follows:

$$|x|^{\frac{n}{\tilde{p}} - \sigma} |u(x)| \lesssim \| |D|^\sigma u \|_{L^p L^{\tilde{p}}}, \quad \frac{n-1}{\tilde{p}} + \frac{1}{p} < \sigma < \frac{n}{p}$$

for arbitrary non radial functions u and all $1 < p < \infty$, $1 \leq \tilde{p} \leq \infty$ (see Subsection 1.1 below for details and more general results).

A central role in our approach will be played by the fractional integrals

$$(T_\gamma \phi)(x) = \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^\gamma} dy, \quad 0 < \gamma < n.$$

Weighted L^p estimates for T_γ are a fundamental problem of harmonic analysis, with a wide range of applications. Starting from the classical one dimensional case studied by Hardy and Littlewood, an exhaustive analysis has been made of the admissible classes of weights and ranges of indices (see [16] and the references therein). In the special case of power weights the optimal result is due to Stein and Weiss:

Theorem 1.1 ([17]). *Let $n \geq 1$ and $1 < p \leq q < \infty$. Assume α, β, γ satisfy the set of conditions $(1 = 1/p + 1/p')$*

$$\begin{aligned} \beta &< \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \gamma < n \\ \alpha + \beta + \gamma &= n + \frac{n}{q} - \frac{n}{p} \\ \alpha + \beta &\geq 0. \end{aligned} \tag{1.2}$$

Then the following inequality holds

$$\| |x|^{-\beta} T_\gamma \phi \|_{L^q} \leq C(\alpha, \beta, p, q) \cdot \| |x|^\alpha \phi \|_{L^p}. \tag{1.3}$$

Conditions in the first line of (1.2) are necessary to ensure integrability, while the necessity of the condition on the second line is due to scaling. On the other hand, the sharpness of $\alpha + \beta \geq 0$ is less obvious and follows from the results of [14].

In the radial case the last condition can be relaxed and $\alpha + \beta$ is allowed to assume negative values. Radial improvements were noticed in [20], [9], and the sharp result was obtained recently by De Napoli, Dreichman and Durán:

Theorem 1.2 ([4]). *Let $n, p, q, \alpha, \beta, \gamma$ be as in the statement of Theorem (1.1) but with the condition $\alpha + \beta \geq 0$ relaxed to*

$$\alpha + \beta \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} \right). \tag{1.4}$$

Then estimate (1.3) is valid for all radial functions $\phi = \phi(|x|)$.

Using the $L_{|x|}^p L_\theta^{\tilde{p}}$ norms we are able prove the following general result which extends both theorems:

Theorem 1.3. *Let $n \geq 2$ and $1 < p \leq q < \infty$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$. Assume α, β, γ satisfy the set of conditions*

$$\begin{aligned} \beta &< \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \gamma < n \\ \alpha + \beta + \gamma &= n + \frac{n}{q} - \frac{n}{p} \\ \alpha + \beta &\geq (n-1) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} \right). \end{aligned} \quad (1.5)$$

Then the following estimate holds:

$$\| |x|^{-\beta} T_\gamma \phi \|_{L^q_{|x|} L^{\tilde{q}}_\theta} \leq C \| |x|^\alpha \phi \|_{L^p_{|x|} L^{\tilde{p}}_\theta}. \quad (1.6)$$

The range of admissible p, q indices can be relaxed to $1 \leq p \leq q \leq \infty$ in two cases:

- (i) when the third inequality in (1.5) is strict, or
- (ii) when the Fourier transform $\hat{\phi}$ has support contained in an annulus $c_1 R \leq |\xi| \leq c_2 R$ ($c_2 \geq c_1 > 0$, $R > 0$); in this case (1.6) holds with a constant independent of R .

Remark 1.1. Notice that:

- (a) with the choices $q = \tilde{q}$ and $p = \tilde{p}$ (i.e. in the usual L^p norms) Theorem 1.3 reduces to Theorem 1.1;
- (b) if ϕ is radially symmetric, with the choice $\tilde{q} = \tilde{p}$, Theorem 1.3 reduces to Theorem 1.2. Indeed, if ϕ is radially symmetric then $T_\gamma \phi$ is radially symmetric too, so that all choices for \tilde{q}, \tilde{p} are equivalent;
- (c) obviously, the same estimate is true for general operators T_F with nonradial kernels $F(x)$ satisfying

$$T_F \phi(x) = \int F(x-y) \phi(y) dy, \quad |F| \leq C |x|^{-\gamma}.$$

The proof of Theorem 1.3 is based on two successive applications of Young's inequality for convolutions on suitable Lie groups: first we use the strong inequality on the rotation group $SO(n)$; then we use a Young inequality in the radial variable, which in some cases must be replaced by the weak Young-Marcinkewicz inequality on the multiplicative group (\mathbb{R}^+, \cdot) with the Haar measure $d\rho/\rho$. The convenient idea of using convolution in the measure $d\rho/\rho$ was introduced in [4].

Remark 1.2. The operator T_γ is a convolution with the homogenous kernel $|x|^{-\gamma}$. Consider instead the convolution with a nonhomogeneous kernel

$$S_\gamma \phi(x) = \int \frac{\phi(y)}{\langle x-y \rangle^\gamma} dy.$$

By the obvious pointwise bound

$$|S_\gamma \phi(x)| \leq T_\gamma |\phi|(x)$$

it is clear that S_γ satisfies the same estimates as T_γ . However the scaling invariance of the estimate is broken, and indeed something more can be proved, thanks to the smoothness of the kernel (see Lemma 2.3):

Corollary 1.4. *Let $n \geq 2$ and $1 \leq p \leq q \leq \infty$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$. Assume α, β, γ satisfy the set of conditions*

$$\beta < \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad \alpha + \beta \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} \right), \quad (1.7)$$

$$\alpha + \beta + \gamma > n \left(1 + \frac{1}{q} - \frac{1}{p} \right). \quad (1.8)$$

Then the following estimate holds:

$$\| |x|^{-\beta} S_\gamma \phi \|_{L^q_{|x|} L^{\tilde{q}}_\theta} \leq C \| |x|^\alpha \phi \|_{L^p_{|x|} L^{\tilde{p}}_\theta}. \quad (1.9)$$

From the basic Theorem 1.3 a large number of inequalities can be deduced, which extend several important classical estimates; we list a few examples in the following.

1.1. Weighted Sobolev embeddings. Recalling the pointwise bound

$$|u(x)| \leq CT_\lambda(|D|^{n-\lambda}u), \quad 0 < \lambda < n \quad (1.10)$$

where $|D|^\sigma = (-\Delta)^{\frac{\sigma}{2}}$, we see that an immediate consequence of (1.9) is the weighted Sobolev inequality

$$\| |x|^{-\beta} u \|_{L^q_{|x|} L^{\tilde{q}}_\theta} \lesssim \| |x|^\alpha |D|^\sigma u \|_{L^p_{|x|} L^{\tilde{p}}_\theta} \quad (1.11)$$

provided $1 < p \leq q < \infty$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$ and

$$\begin{aligned} \beta &< \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \sigma < n \\ \alpha + \beta &= \sigma + \frac{n}{q} - \frac{n}{p} \\ \alpha + \beta &\geq (n-1) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} \right). \end{aligned} \quad (1.12)$$

As usual, if the last condition is strict we can take p, q in the full range $1 \leq p \leq q \leq \infty$. For instance, this implies the inequality

$$|x|^{-\beta} |u(x)| \lesssim \| |x|^\alpha |D|^\sigma u \|_{L^p_{|x|} L^{\tilde{p}}_\theta} \quad (1.13)$$

provided $1 \leq p \leq \infty$ and

$$\begin{aligned} \beta &< 0, \quad \alpha < \frac{n}{p'}, \quad 0 < \sigma < n \\ \alpha + \beta &= \sigma - \frac{n}{p} \\ \alpha + \beta &> (n-1) \left(\frac{1}{\tilde{p}} - \frac{1}{p} \right). \end{aligned}$$

If we choose $\alpha = 0$ we have in particular for $p \in (1, \infty)$, $\tilde{p} \in [1, \infty]$

$$|x|^{\frac{n}{p}-\sigma} |u(x)| \lesssim \| |D|^\sigma u \|_{L^p_{|x|} L^{\tilde{p}}_\theta}, \quad \frac{n-1}{\tilde{p}} + \frac{1}{p} < \sigma < \frac{n}{p}. \quad (1.14)$$

This extends to the non radial case the radial inequalities in [19], [13], [2] and many others; notice that in the radial case we can choose $\tilde{p} = \infty$ to obtain the largest possible range. When σ is an integer we can replace the fractional operator $|D|^\sigma$ with usual derivatives; see Corollary 1.9 below for a similar argument.

By similar techniques it is possible to derive nonhomogeneous estimates in terms of norms of type $\| \langle D \rangle^\sigma u \|_{L^p}$; we omit the details.

1.2. Critical estimates in Besov spaces. Case (ii) in Theorem 1.3 is suitable for applications to spaces defined via Fourier decompositions, in particular Besov spaces. We recall the standard machinery: fix a C_c^∞ radial function $\psi_0(\xi)$ equal to 1 for $|\xi| < 1$ and vanishing for $|\xi| > 2$, define a Littlewood-Paley partition of unity via $\phi_0(\xi) = \psi(\xi) - \psi(\xi/2)$, $\phi_j(\xi) = \phi_0(2^{-j}\xi)$, and decompose u as $u = \sum_{j \in \mathbb{Z}} u_j$ where $u_j = \phi_j(D)u = \mathcal{F}^{-1} \phi_j(\xi) \mathcal{F} u$. Then the homogeneous Besov norm $\dot{B}_{p,1}^s$ is defined as

$$\|u\|_{\dot{B}_{p,1}^s} = \sum_{j \in \mathbb{Z}} 2^{js} \|u_j\|_{L^p}. \quad (1.15)$$

We can apply Theorem 1.3-(ii) to each component u_j in the full range of indices $1 \leq p \leq q \leq \infty$, with a constant independent of j . By the standard trick $\tilde{u}_j = u_{j-1} + u_j + u_{j+1}$, $u_j = \phi_j(D)\tilde{u}_j$ we obtain the estimate

$$\| |x|^{-\beta} T_\gamma u \|_{L^q_{|x|} L^{\tilde{q}}_\theta} \leq C \sum_{j \in \mathbb{Z}} \| |x|^\alpha \tilde{u}_j \|_{L^p_{|x|} L^{\tilde{p}}_\theta} \quad (1.16)$$

for the full range $1 \leq p \leq q \leq \infty$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$, with α, β, γ satisfying (1.12). The right hand side can be interpreted as a weighted norm of Besov type with different radial and angular integrability; in the special case $\alpha = 0$, $p = \tilde{p} > 1$ we obtain a standard Besov norm (1.15) and hence the estimate (with the optimal choice $\tilde{q} = \tilde{p} = p$) reduces to

$$\| |x|^{-\beta} T_\gamma u \|_{L^q_{|x|} L^p_\theta} \leq C \|u\|_{\dot{B}^0_{p,1}}. \quad (1.17)$$

This estimate is weaker than (1.9) when the third condition in (1.7) is strict, but in the case of equality it gives a new estimate: recalling (1.10), we have proved the following

Corollary 1.5. *For all $1 < p \leq q \leq \infty$ we have*

$$\| |x|^{\frac{n-1}{p} - \frac{n-1}{q}} u \|_{L^q_{|x|} L^p_\theta} \leq C \|u\|_{\dot{B}^{\frac{1}{p} - \frac{1}{q}}_{p,1}}. \quad (1.18)$$

If we restrict (1.18) to radial functions and $q = \infty$, we obtain the well known radial pointwise estimate

$$|x|^{\frac{n-1}{p}} |u| \leq C \|u\|_{\dot{B}^{1/p}_{p,1}} \quad 1 < p < \infty \quad (1.19)$$

(see [2], [15]).

1.3. Caffarelli-Kohn-Nirenberg weighted interpolation inequalities. Consider the family of inequalities on \mathbb{R}^n , $n \geq 1$

$$\| |x|^{-\gamma} u \|_{L^r} \leq C \| |x|^{-\alpha} \nabla u \|_{L^p}^a \| |x|^{-\beta} u \|_{L^q}^{1-a}. \quad (1.20)$$

for the range of parameters

$$n \geq 1, \quad 1 \leq p < \infty, \quad 1 \leq q < \infty, \quad 0 < r < \infty, \quad 0 < a \leq 1. \quad (1.21)$$

Some conditions are immediately seen to be necessary for the validity of (1.20): to ensure local integrability we need

$$\gamma < \frac{n}{r} \quad \alpha < \frac{n}{p} \quad \beta < \frac{n}{q} \quad (1.22)$$

and by scaling invariance we need to assume

$$\gamma - \frac{n}{r} = a \left(\alpha + 1 - \frac{n}{p} \right) + (1-a) \left(\beta - \frac{n}{q} \right). \quad (1.23)$$

In [1] the following remarkable result was proved, which improves and extends a number of earlier estimates including weighted Sobolev and Hardy inequalities:

Theorem 1.6 ([1]). *Consider the inequalities (1.20) in the range of parameters given by (1.22), (1.21), (1.23). Denote with Δ the quantity*

$$\Delta = \gamma - a\alpha - (1-a)\beta \equiv a + n \left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p} \right) \quad (1.24)$$

(the identity in (1.24) is a reformulation of the scaling relation (1.23)). Then the inequalities (1.20) are true if and only if both the following conditions are satisfied:

- (i) $\Delta \geq 0$
- (ii) $\Delta \leq a$ when $\gamma - n/r = \alpha + 1 - n/p$.

Remark 1.3. Notice that in the original formulation of [1] also the case $a = 0$ was considered, but with the introduction of an additional parameter forcing $\beta = \gamma$ when $a = 0$. Thus the case $a = 0$ becomes trivial in the original formulation; however, at least for $r > 1$, a much larger range $0 \leq \gamma - \beta < n$ can be obtained by a direct application of the Hardy-Littlewood-Sobolev inequality, so strictly speaking the additional requirement $\beta = \gamma$ is not necessary. We think the formulation adopted here is cleaner.

On the other hand, the necessity of (i) follows from the uniformity of the estimate w.r.to translations, while the necessity of (ii) is proved by testing the inequality on the spikes $|x|^{\gamma-n/r} \log |x|^{-1}$ truncated near $x = 0$.

In [5] the authors prove the following radial improvement of Theorem 1.6:

Theorem 1.7 ([5]). *Let $n \geq 2$, let $\alpha, \beta, \gamma, r, p, q, a$ be in the range determined by (1.22), (1.21), (1.23), define Δ as in (1.24), and assume that*

$$a \left(1 - \frac{n}{p}\right) \leq \Delta \leq a, \quad \alpha < \frac{n}{p} - 1, \quad (1.25)$$

the first inequality being strict when $p = 1$. Then estimate (1.20) is true for all radial functions $u \in C_c^\infty(\mathbb{R}^n)$.

We somewhat simplified the statement of Theorem 1.1 in [5], and in particular conditions (1.8)-(1.10) in that paper are equivalent to (1.25) here, as it is readily seen. Notice that the condition $\Delta \leq a$ forces r to be larger than 1.

Using the $L^p L^{\tilde{p}}$ norms we can extend both Theorems 1.6 and 1.7. For greater generality we prove an estimate with fractional derivatives

$$|D|^\sigma = (-\Delta)^{\frac{\sigma}{2}}, \quad \sigma > 0.$$

Our result is the following:

Theorem 1.8. *Let $n \geq 2$, $r, \tilde{r}, p, \tilde{p}, q, \tilde{q} \in [1, +\infty)$, $0 < a \leq 1$, $0 < \sigma < n$ with*

$$\gamma < \frac{n}{r}, \quad \beta < \frac{n}{q}, \quad \frac{n}{p} - n < \alpha < \frac{n}{p} - \sigma \quad (1.26)$$

satisfying the scaling condition

$$\gamma - \frac{n}{r} = a \left(\alpha + \sigma - \frac{n}{r} \right) + (1-a) \left(\beta - \frac{n}{q} \right). \quad (1.27)$$

Define the quantities

$$\Delta = a\sigma + n \left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p} \right), \quad \tilde{\Delta} = a\sigma + n \left(\frac{1}{\tilde{r}} - \frac{1-a}{\tilde{q}} - \frac{a}{\tilde{p}} \right). \quad (1.28)$$

and assume further that

$$\Delta + (n-1)\tilde{\Delta} \geq 0, \quad (1.29)$$

$$1 < p, \quad a \left(\sigma - \frac{n}{p} \right) < \Delta \leq a\sigma, \quad a \left(\sigma - \frac{n}{\tilde{p}} \right) \leq \tilde{\Delta} \leq a\sigma. \quad (1.30)$$

Then the following interpolation inequality holds:

$$\| |x|^{-\gamma} u \|_{L_{|x|}^r L_{\theta}^{\tilde{r}}} \leq C \| |x|^{-\alpha} |D|^\sigma u \|_{L_{|x|}^p L_{\theta}^{\tilde{p}}}^a \| |x|^{-\beta} u \|_{L_{|x|}^q L_{\theta}^{\tilde{q}}}^{1-a}. \quad (1.31)$$

If one assumes strict inequality in (1.29), then the inequalities in (1.30) can be relaxed to non strict inequalities.

When σ is an integer, the condition on α from below can be dropped, and a slightly stronger estimate can be proved. We introduce the notation

$$\| |x|^{-\alpha} D^\sigma u \|_{L^p L^{\tilde{p}}} = \sum_{|\nu|=\sigma} \| |x|^{-\alpha} D^\nu u \|_{L^p L^{\tilde{p}}}, \quad \nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n.$$

Then we have:

Corollary 1.9. *Assume $\sigma = 1, \dots, n-1$ is an integer. Then the following estimate holds*

$$\| |x|^{-\gamma} u \|_{L_{|x|}^r L_{\theta}^{\tilde{r}}} \leq C \| |x|^{-\alpha} D^{\sigma} u \|_{L_{|x|}^p L_{\theta}^{\tilde{p}}}^a \| |x|^{-\beta} u \|_{L_{|x|}^q L_{\theta}^{\tilde{q}}}^{1-a}. \quad (1.32)$$

provided the parameters satisfy the same conditions as in the previous theorem, with the exception of the condition $\alpha > -n + n/p$ which is not necessary.

Remark 1.4. If $\sigma = 1$, Corollary 1.9 contains both the original result of [1] (for $\Delta \leq a$) and the radial improvement of [4].

Indeed, if we choose $p = \tilde{p}$, $q = \tilde{q}$, $r = \tilde{r}$ in Corollary 1.9 we get of course $\Delta = \tilde{\Delta}$, and selecting $\sigma = 1$ we reobtain the original inequality (1.20) in the range $0 \leq \Delta \leq a$.

On the other hand, if u is a radial function, estimate (1.32) does not depend on the choice of $\tilde{p}, \tilde{q}, \tilde{r}$ and we can let $\tilde{\Delta}$ assume an arbitrary value in the range (1.30). Thus if $\Delta > a(\sigma - n/p)$ we can choose $\tilde{\Delta} = 0$, while if $\Delta = a(\sigma - n/p)$ we can choose $\tilde{\Delta} = \epsilon > 0$ arbitrarily small, recovering the results of Theorem 1.7.

The classical application of CKN estimates is to the regularity of solutions to the Navier-Stokes equation; this will be the subject of forthcoming papers.

1.4. Strichartz estimates for the wave equation. As a last example, we mention an application of our result to Strichartz estimates for the wave equation; a more detailed analysis will be conducted elsewhere. The wave flow $e^{it|D|}$ on \mathbb{R}^n , $n \geq 2$, satisfies the estimates, which are usually called *Strichartz estimates*:

$$\| |D|^{\frac{n}{r} + \frac{1}{p} - \frac{n}{2}} e^{it|D|} f \|_{L_t^p L_x^r} \lesssim \| f \|_{L^2} \quad (1.33)$$

provided the indices p, r satisfy

$$p \in [2, \infty], \quad 0 < \frac{1}{r} \leq \frac{1}{2} - \frac{2}{(n-1)p}. \quad (1.34)$$

Here the $L_t^p L_x^r$ norms are defined as

$$\| u(t, x) \|_{L_t^p L_x^r} = \| \| u(t, \cdot) \|_{L_x^r} \|_{L_t^p}.$$

In their most general version, the estimates were proved in [7], [11]. Notice that in (1.33) we included the extension of the estimates which can be obtained via Sobolev embedding on \mathbb{R}^n .

If the initial value f is a radial function, the estimates admit an improvement in the sense that conditions (1.34) can be relaxed to

$$p \in [2, \infty], \quad 0 < \frac{1}{r} < \frac{1}{2} - \frac{1}{(n-1)p}. \quad (1.35)$$

This phenomenon is connected with the finite speed of propagation for the wave equation and is usually deduced using the space-time decay properties of the equation. For a thorough discussion and a comprehensive history of such estimates see e.g. [10] and the references therein.

A different set of estimates are the *smoothing estimates*, also known as Morawetz-type or weak dispersion estimates. These appear in a large number of versions; a particularly sharp one is the following, from [6]:

$$\| |x|^{-\zeta} |D|^{\frac{1}{2}-\zeta} e^{it|D|} f \|_{L_t^2 L_x^2} \lesssim \| \Lambda^{\frac{1}{2}-\zeta} f \|_{L^2}, \quad \frac{1}{2} < \zeta < \frac{n}{2}. \quad (1.36)$$

Here the operator

$$\Lambda = (1 - \Delta_{\mathbb{S}^{n-1}})^{1/2}$$

is a function of the Laplace-Beltrami operator on the sphere and acts only on angular variables, thus we see that the flow improves the angular regularity. Morawetz-type estimates are conceptually simpler than (1.33), being related to more basic properties of the operators; indeed L^2 estimates of this type can be proved for quite large classes of equations via multiplier methods.

Corresponding estimates are known for the Schrödinger flow $e^{it\Delta}$, and M.C. Vilela [20] noticed that in the radial case they can be used to deduce Strichartz estimates via the radial Sobolev embedding. Following a similar idea for the wave flow, in combination with our precised estimates (1.11), gives an even better result, which strengthens the standard Strichartz estimates (1.33)-(1.34) in terms of the mixed $L_{|x|}^p L_{\theta}^{\tilde{p}}$ norms. Indeed, a special case of (1.11) gives, for arbitrary functions $g(x)$,

$$\|g\|_{L_{|x|}^q L_{\theta}^{\tilde{q}}} \lesssim \| |x|^{\alpha} |D|^{\alpha + \frac{n}{2} - \frac{n}{q}} g \|_{L^2}, \quad q, \tilde{q} \in [2, \infty), \quad \frac{n}{2} > \alpha \geq (n-1) \left(\frac{1}{q} - \frac{1}{\tilde{q}} \right) \quad (1.37)$$

with the exclusion of the case $\alpha = 0$, $q = \tilde{q} = 2$. Then by (1.37) and (1.36) we obtain the precised Strichartz estimates

$$\| |x|^{-\delta} |D|^{\frac{n}{q} + \frac{1}{2} - \frac{n}{2} - \delta} e^{it|D|} f \|_{L_t^2 L_{|x|}^q L_{\theta}^{\tilde{q}}} \lesssim \|\Lambda^{-\epsilon} f\|_{L^2} \quad (1.38)$$

provided

$$q, \tilde{q} \in [2, +\infty), \quad \delta < \frac{n}{q}, \quad 0 < \epsilon < \frac{n-1}{2}, \quad 0 < \frac{1}{q} < \frac{1}{\tilde{q}} - \frac{1}{2(n-1)} \quad (1.39)$$

and

$$\epsilon \leq \delta + (n-1) \left(\frac{1}{\tilde{q}} - \frac{1}{2(n-1)} - \frac{1}{q} \right). \quad (1.40)$$

2. PROOF OF THEOREM 1.3

The first result we need is an explicit estimate of the angular part of the fractional integral $T_{\gamma}\phi$. Notice that a similar analysis in the radial case was done in [4] (see Lemma 4.2 there). The following estimates are sharp:

Lemma 2.1. *Let $n \geq 2$, $\nu > 0$, and write $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then the integral*

$$I_{\nu}(x) = \int_{\mathbb{S}^{n-1}} |x - y|^{-\nu} dS(y) \quad x \in \mathbb{R}^n$$

satisfies

$$|I_{\nu}(x)| \simeq \langle x \rangle^{-\nu} \quad \text{for } |x| \geq 2, \quad (2.1)$$

while for $|x| \leq 2$ we have

$$|I_{\nu}(x)| \simeq \begin{cases} 1 & \text{if } \nu < n-1 \\ |\log ||x| - 1|| + 1 & \text{if } \nu = n-1 \\ ||x| - 1|^{n-1-\nu} & \text{if } \nu > n-1. \end{cases} \quad (2.2)$$

Proof. We consider four different regimes according to the size of $|x|$. We write for brevity I instead of I_{ν} .

First case: $|x| \geq 2$. For x large and $|y| = 1$ we have $|x - y| \simeq |x|$, hence $|I(x)| \simeq |x|^{-\nu} \simeq \langle x \rangle^{-\nu}$. This proves (2.1).

Second case: $0 \leq |x| \leq \frac{1}{2}$. Clearly we have $|x - y| \simeq 1$ when $|y| = 1$, and this implies $|I(x)| \simeq 1 \simeq \langle x \rangle^{-\nu}$. This is equivalent to (2.2) when $|x| \leq 1/2$.

Third case: $1 \leq |x| \leq 2$. This is the bulk of the computation since it contains the singular part of the integral, as $|x| \rightarrow 1$. We write the integral in polar coordinates using the spherical angles $(\theta_1, \theta_2, \dots, \theta_{n-1})$ on \mathbb{S}^{n-1} , oriented in such a way that θ_1 is the angle between x and y . Using the notation $\sigma = |x - y|$, by the symmetry of $I(x)$ in $(\theta_2, \dots, \theta_{n-1})$ we have

$$|I(x)| \simeq \int_0^\pi \sigma^{-\nu} (\sin \theta_1)^{n-2} d\theta_1.$$

In order to rewrite the integral using σ as a new variable, we compute

$$2\sigma d\sigma = d(|x - y|^2) = d(|x|^2 + |y|^2 - 2|x||y|\cos \theta_1) = 2|x| \sin \theta_1 d\theta_1$$

so we have

$$(\sin \theta_1)^{n-2} d\theta_1 = \frac{\sigma (\sin \theta_1)^{n-3}}{|x|} d\sigma$$

and, noticing that $0 \leq |x| - 1 \leq |x - y| = \sigma \leq |x| + 1$,

$$|I(x)| \simeq \int_{|x|-1}^{|x|+1} \sigma^{1-\nu} \frac{(\sin \theta_1)^{n-3}}{|x|} d\sigma.$$

Now let A be the area of the triangle with vertices $0, x$ and y : we have $2A = |x| \sin \theta_1$ so that

$$|I(x)| \simeq |x|^{2-n} \int_{|x|-1}^{|x|+1} \sigma^{1-\nu} A^{n-3} d\sigma.$$

Recalling Heron's formula for the area of a triangle as a function of the length of its sides we obtain

$$|I(x)| \simeq |x|^{2-n} \int_{|x|-1}^{|x|+1} \sigma^{1-\nu} \left[(|x| + \sigma + 1)(|x| + \sigma - 1)(|x| + 1 - \sigma)(\sigma + 1 - |x|) \right]^{\frac{n-3}{2}} d\sigma.$$

Notice that this formula is correct for all dimensions $n \geq 2$.

Now we split the integral as $I \simeq I_1 + I_2$ with

$$I_1(x) = |x|^{2-n} \int_{|x|-1}^{|x|} \sigma^{1-\nu} \left[(|x| + \sigma + 1)(|x| + \sigma - 1)(|x| + 1 - \sigma)(\sigma + 1 - |x|) \right]^{\frac{n-3}{2}} d\sigma$$

and

$$I_2(x) = |x|^{2-n} \int_{|x|}^{|x|+1} \sigma^{1-\nu} \left[(|x| + \sigma + 1)(|x| + \sigma - 1)(|x| + 1 - \sigma)(\sigma + 1 - |x|) \right]^{\frac{n-3}{2}} d\sigma.$$

In the second integral I_2 , recalling that $1 \leq |x| \leq 2$, we have

$$|x| \simeq \sigma \simeq |x| + \sigma + 1 \simeq |x| + \sigma - 1 \simeq \sigma + 1 - |x| \simeq 1$$

so that

$$I_2 \simeq \int_{|x|}^{|x|+1} (|x| + 1 - \sigma)^{\frac{n-3}{2}} d\sigma = \int_0^1 (1 - \sigma)^{\frac{n-3}{2}} d\sigma \simeq 1.$$

In the first integral I_1 , using that $1 \leq |x| \leq 2$ and $|x| - 1 \leq \sigma \leq |x|$, we see that

$$|x| \simeq (|x| + \sigma + 1) \simeq (|x| + 1 - \sigma) \simeq 1;$$

moreover,

$$1 \leq \frac{|x| + \sigma - 1}{\sigma} \leq 2 \quad \text{so that} \quad |x| + \sigma - 1 \simeq \sigma$$

and we have

$$I_1(x) \simeq \int_{|x|-1}^{|x|} \sigma^{1-\nu+\frac{n-3}{2}} (\sigma + 1 - |x|)^{\frac{n-2}{2}} d\sigma$$

or, after the change of variable $\sigma \rightarrow \sigma(|x| - 1)$,

$$I_1(x) \simeq (|x| - 1)^{n-1-\nu} \int_1^{1+\frac{1}{|x|-1}} (\sigma - 1)^{\frac{n-3}{2}} \sigma^{\frac{n-1}{2}-\nu} d\sigma.$$

Now split the last integral as $A + B$ where

$$A = (|x| - 1)^{n-1-\nu} \int_1^2 (\sigma - 1)^{\frac{n-3}{2}} \sigma^{\frac{n-1}{2}-\nu} d\sigma$$

and

$$B = (|x| - 1)^{n-1-\nu} \int_2^{1+\frac{1}{|x|-1}} (\sigma - 1)^{\frac{n-3}{2}} \sigma^{\frac{n-1}{2}-\nu} d\sigma;$$

we have immediately

$$A \simeq (|x| - 1)^{n-1-\nu}$$

while, keeping into account that $\sigma \simeq \sigma - 1$ for σ in $(2, 1 + \frac{1}{|x|-1})$,

$$B = (|x| - 1)^{n-1-\nu} \int_2^{1+\frac{1}{|x|-1}} \sigma^{n-2-\nu} d\sigma$$

which gives

$$B \simeq \begin{cases} 1 & \text{if } \nu < n - 1 \\ |\log ||x| - 1|| + 1 & \text{if } \nu = n - 1 \\ ||x| - 1|^{n-1-\nu} & \text{if } \nu > n - 1 \end{cases} \quad (2.3)$$

Fourth case: $\frac{1}{2} \leq |x| \leq 1$. Using the change of variable $|x'| = 1/|x|$, we see that $|I(x)| \simeq |I(1/|x'|)|$, thus the fourth case follows immediately from the third one, and this concludes the proof of the Lemma. \square

We shall also need the following estimate which is proved in a similar way:

Lemma 2.2. *Let $n \geq 2$, $\nu > 0$. Then the integral*

$$J_\nu(x, \rho) = \int_{\mathbb{S}^{n-1}} \langle x - \rho\theta \rangle^{-\nu} dS(\theta) \quad x \in \mathbb{R}^n, \rho \geq 0$$

satisfies:

$$|J_\nu(x, \rho)| \simeq \langle x \rangle^{-\nu} \quad \text{for } \rho \leq 1 \text{ or } |x| \geq 2\rho, \quad (2.4)$$

$$|J_\nu(x, \rho)| \simeq \langle \rho \rangle^{-\nu} \quad \text{for } |x| \leq 1 \text{ or } \rho \geq 2|x|, \quad (2.5)$$

while in the remaining case, i.e. when $|x| \geq 1$ and $\rho \geq 1$ and $2^{-1}|x| \leq \rho \leq 2|x|$,

$$|J_\nu(x, \rho)| \simeq \begin{cases} \langle \rho \rangle^{-\nu} & \text{if } \nu < n - 1 \\ \langle \rho \rangle^{-\nu} \log \left(\frac{2\langle \rho \rangle}{\langle |x| - \rho \rangle} \right) & \text{if } \nu = n - 1 \\ \langle \rho \rangle^{1-n} \langle |x| - \rho \rangle^{n-1-\nu} & \text{if } \nu > n - 1. \end{cases} \quad (2.6)$$

As a consequence, one has $J_\nu \lesssim \langle \rho + |x| \rangle^{-\nu}$ when $\nu < n - 1$ and $J_\nu \lesssim \langle \rho + |x| \rangle^{-\nu} \log(2\langle \rho \rangle + |x|)$ when $\nu = n - 1$.

Proof. The proof is similar to the proof of Lemma 2.1; we sketch the main steps. Estimates (2.4) and (2.5) are obvious, thus we focus on (2.6). Write $r = |x|$, so that we are in the region $1/2 \leq r/\rho \leq 2$; we shall consider in detail the case

$$1 \leq \frac{r}{\rho} \leq 2,$$

the remaining region being similar. Using the same coordinates as before, the integral is reduced to

$$J_\nu(|x|, \rho) = |x|^{2-n} \int_{|x|-1}^{|x|+1} \langle \rho\sigma \rangle^{-\nu} A^{n-3} \sigma d\sigma$$

where A is given by Heron's formula

$$A(|x|, \sigma)^2 = (|x| + \sigma + 1)(|x| + \sigma - 1)(|x| + 1 - \sigma)(\sigma + 1 - |x|).$$

We split the integral on the intervals $|x| \leq \sigma \leq |x| + 1$ and $|x| - 1 \leq \sigma \leq |x|$. The first piece gives

$$I_1 \simeq \langle \rho \rangle^{-\nu} \int_{|x|}^{|x|+1} (|x| + 1 - \sigma)^{\frac{n-3}{2}} d\sigma$$

and by the change of variable $\sigma \rightarrow \sigma(|x| + 1)$ we obtain

$$I_1(|x|, \rho) \simeq \langle \rho \rangle^{-\nu}.$$

For the second integral on $|x| - 1 \leq \sigma \leq |x|$, noticing that

$$1 \leq \frac{|x| + \sigma - 1}{\sigma} \leq 2$$

we have

$$\begin{aligned} I_2 &\simeq \int_{|x|-1}^{|x|} \langle \rho \sigma \rangle^{-\nu} \sigma^{\frac{n-1}{2}} (\sigma + 1 - |x|)^{\frac{n-3}{2}} d\sigma \\ &= (|x| - 1)^{n-1} \int_1^{\frac{|x|}{|x|-1}} \langle (r - \rho) \sigma \rangle^{-\nu} \sigma^{\frac{n-1}{2}} (\sigma - 1)^{\frac{n-3}{2}} d\sigma \end{aligned}$$

via the change of variables $\sigma \rightarrow \sigma(|x| - 1)$ which gives $\rho \sigma \rightarrow (r - \rho) \sigma$. The part of the integral between 1 and 2 produces

$$\simeq (|x| - 1)^{n-1} \langle r - \rho \rangle^{-\nu} = \rho^{1-n} (r - \rho)^{n-1} \langle r - \rho \rangle^{-\nu}$$

while the remaining part between 2 and $|x|/(|x| - 1)$ gives

$$\begin{aligned} &\simeq (|x| - 1)^{n-1} \int_2^{\frac{r}{r-\rho}} \langle (r - \rho) \sigma \rangle^{-\nu} \sigma^{n-2} d\sigma \\ &= \rho^{1-n} \int_{2(r-\rho)}^r \langle \sigma \rangle^{-\nu} \sigma^{n-2} d\sigma \\ &\simeq \rho^{1-n} \int_{2(r-\rho)}^r \frac{\sigma^{n-2}}{1 + \sigma^\nu} d\sigma \end{aligned}$$

which can be computed explicitly. Summing up we obtain (2.6). \square

We are ready for the main part of the proof. By the isomorphism

$$\mathbb{S}^{n-1} \simeq SO(n)/SO(n-1)$$

we can represent integrals on \mathbb{S}^{n-1} in the form

$$\int_{\mathbb{S}^{n-1}} g(y) dS(y) = c_n \int_{SO(n)} g(Ae) dA, \quad n \geq 2$$

where dA is the left Haar measure on $SO(n)$, and $e \in \mathbb{S}^{n-1}$ is a fixed arbitrary unit vector. Thus, via polar coordinates, a convolution integral can be written as follows (apart from inessential constants depending only on the space dimension n):

$$\begin{aligned} F * \phi(x) &= \int_{\mathbb{R}^n} F(x - y) \phi(y) dy = \int_0^\infty \int_{\mathbb{S}^{n-1}} F(x - \rho \omega) \phi(\rho \omega) dS_\omega \rho^{n-1} d\rho \\ &\simeq \int_0^\infty \int_{SO(n)} F(x - \rho B e) \phi(\rho B e) dB \rho^{n-1} d\rho \end{aligned}$$

Hence the $L^{\tilde{q}}$ norm of the convolution on the sphere can be written as

$$\begin{aligned} \|F * \phi(|x|\theta)\|_{L^{\tilde{q}}_{\theta}(\mathbb{S}^{n-1})} &\simeq \|F * \phi(|x|Ae)\|_{L^{\tilde{q}}_A(SO(n))} \\ &\leq \int_0^\infty \left\| \int_{SO(n)} F(|x|Ae - \rho Be)\phi(\rho Be)dB \right\|_{L^{\tilde{q}}_A(SO(n))} \rho^{n-1}d\rho \end{aligned}$$

where e is any fixed unit vector. By the change of variables $B \rightarrow AB^{-1}$ in the inner integral (and the invariance of the measure) this is equivalent to

$$= \int_0^\infty \left\| \int_{SO(n)} F(AB^{-1}(|x|Be - \rho e))\phi(\rho AB^{-1}e)dB \right\|_{L^{\tilde{q}}_A(SO(n))} \rho^{n-1}d\rho$$

If F satisfies

$$|F(x)| \leq Cf(|x|) \quad (2.7)$$

for a radial function f , we can write

$$|F(AB^{-1}(|x|Be - \rho e))| \leq Cf(|x|Be - \rho e|)$$

and we notice that the integral

$$\int_{SO(n)} f(|x|Be - \rho e|) |\phi(\rho AB^{-1}e)|dB = g * h(A)$$

is a convolution on $SO(n)$ of the functions

$$g(A) = f(|x|Ae - \rho e|), \quad h(A) = |\phi(\rho Ae)|.$$

We can thus apply the Young's inequality on $SO(n)$ (see e.g. Theorem 1.2.12 in [8]) and we obtain, for any

$$\tilde{q}, \tilde{r}, \tilde{p} \in [1, +\infty] \quad \text{with} \quad 1 + \frac{1}{\tilde{q}} = \frac{1}{\tilde{r}} + \frac{1}{\tilde{p}},$$

the estimate

$$\|F * \phi(|x|\theta)\|_{L^{\tilde{q}}_{\theta}(\mathbb{S}^{n-1})} \lesssim \int_0^\infty \|f(|x|e - \rho\theta)\|_{L^{\tilde{r}}_{\theta}(\mathbb{S}^{n-1})} \|\phi(\rho\theta)\|_{L^{\tilde{p}}_{\theta}(\mathbb{S}^{n-1})} \rho^{n-1}d\rho \quad (2.8)$$

where we switched back to the coordinates of \mathbb{S}^{n-1} . Notice that the conditions on the indices imply in particular

$$\tilde{q} \geq \tilde{p}.$$

Specializing f to the choice

$$f(|x|) = |x|^{-\gamma}$$

we get

$$\|F * \phi(|x|\theta)\|_{L^{\tilde{q}}_{\theta}(\mathbb{S}^{n-1})} \lesssim \int_0^\infty \rho^{-\gamma} \|\rho^{-1}|x|e - \theta|^{-\gamma}\|_{L^{\tilde{r}}_{\theta}(\mathbb{S}^{n-1})} \|\phi(\rho\theta)\|_{L^{\tilde{p}}_{\theta}(\mathbb{S}^{n-1})} \rho^{n-1}d\rho$$

which can be written in the form

$$= |x|^{n-\alpha-\frac{n}{p}-\gamma} \int_0^\infty \left(\frac{|x|}{\rho}\right)^{\alpha+\frac{n}{p}-n+\gamma} \|\rho^{-1}|x|e - \theta|^{-\gamma}\|_{L^{\tilde{r}}_{\theta}(\mathbb{S}^{n-1})} \rho^{\alpha+\frac{n}{p}} \|\phi(\rho\theta)\|_{L^{\tilde{p}}_{\theta}(\mathbb{S}^{n-1})} \frac{d\rho}{\rho}$$

or equivalently, recalling (1.2),

$$= |x|^{\beta-\frac{n}{q}} \int_0^\infty \left(\frac{|x|}{\rho}\right)^{-\beta+\frac{n}{q}} \|\rho^{-1}|x|e - \theta|^{-\gamma}\|_{L^{\tilde{r}}_{\theta}(\mathbb{S}^{n-1})} \rho^{\alpha+\frac{n}{p}} \|\phi(\rho\theta)\|_{L^{\tilde{p}}_{\theta}(\mathbb{S}^{n-1})} \frac{d\rho}{\rho}$$

Following [4], we recognize that the last integral is a convolution in the multiplicative group (\mathbb{R}, \cdot) with the Haar measure $d\rho/\rho$, which implies

$$|x|^{-\beta+\frac{n}{q}} \|F * \phi(|x|\theta)\|_{L^{\tilde{q}}_{\theta}(\mathbb{S}^{n-1})} \lesssim g_1 * h_1(|x|),$$

with

$$g_1(\rho) = \rho^{-\beta + \frac{n}{q}} \|\rho e - \theta\|_{L_{\theta}^{\tilde{r}}}, \quad h_1(\rho) = \rho^{\alpha + \frac{n}{p}} \|\phi(\rho\theta)\|_{L_{\theta}^{\tilde{p}}}.$$

By the weak Young's inequality in the measure $d\rho/\rho$ (Theorem 1.4.24 in [8]) we obtain

$$\begin{aligned} \||x|^{-\beta} F * \phi\|_{L^q L^{\tilde{q}}} &\equiv \||x|^{-\beta + \frac{n}{q}} \|F * \phi(|x|\theta)\|_{L_{\theta}^{\tilde{q}}}\|_{L^q(\rho^{-1} d\rho)} \\ &\lesssim \|h_1\|_{L^p(\rho^{-1} d\rho)} \|g_1\|_{L^{r,\infty}(\rho^{-1} d\rho)} \end{aligned}$$

that is to say

$$\||x|^{-\beta} F * \phi\|_{L^q L^{\tilde{q}}} \lesssim \|\phi\|_{L^p L^{\tilde{p}}} \|\rho^{-\beta + \frac{n}{q}} \|\rho e - \theta\|_{L_{\theta}^{\tilde{r}}}\|_{L^{r,\infty}(\rho^{-1} d\rho)}. \quad (2.9)$$

provided

$$q, r, p \in (1, +\infty) \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

In particular this implies

$$q > p. \quad (2.10)$$

In order to achieve the proof, it remains to check that the last norm in (2.9) is finite. Notice that, when $\tilde{r} < \infty$,

$$\|\rho e - \theta\|_{L_{\theta}^{\tilde{r}}}^{-\gamma} = I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}}$$

where I_{ν} was defined and estimated in Lemma 2.1. On the other hand, when $\tilde{r} = \infty$ one has directly

$$\tilde{r} = \infty \implies \|\rho e - \theta\|_{L_{\theta}^{\tilde{r}}}^{-\gamma} \simeq |\rho - 1|^{-\gamma}. \quad (2.11)$$

Using cutoffs, we split the $L^{r,\infty}$ norm in three regions $0 \leq \rho \leq 1/2$, $\rho \geq 2$ and $1/2 \leq \rho \leq 2$.

In the region $0 \leq \rho \leq 1/2$, recalling (2.1)-(2.2) or (2.11), we have

$$I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq 1 \implies \rho^{-\beta + \frac{n}{q}} I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \in L^1(0, 1/2; d\rho/\rho)$$

since by assumption $\beta < n/q$; thus the contribution of this part to the $L^{r,\infty}(d\rho/\rho)$ norm is finite.

In the region $\rho \geq 2$ we have

$$I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq \rho^{-\gamma} \implies \rho^{-\beta + \frac{n}{q}} I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq \rho^{-\beta - \gamma + \frac{n}{q}} \in L^1(2, \infty; d\rho/\rho)$$

since the condition

$$-\beta - \gamma + \frac{n}{q} < 0 \iff \alpha < \frac{n}{p'}$$

is satisfied by (1.7), and again the contribution to the $L^{r,\infty}$ norm is finite.

For the third region $1/2 \leq \rho \leq 2$, by estimate (2.2), we see that in the case $\gamma\tilde{r} \leq n - 1$ one has again, for some $\sigma \geq 0$,

$$I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq |\log \|\rho\| - 1|^{\sigma} \implies \rho^{-\beta + \frac{n}{q}} I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \in L^1(1/2, 2; d\rho/\rho)$$

On the other hand, in the case $\gamma\tilde{r} > n - 1$ (which includes the choice $\tilde{r} = \infty$), we see that

$$\rho^{-\beta + \frac{n}{q}} I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq |\rho - 1|^{\frac{n-1}{\tilde{r}} - \gamma} \in L^{r,\infty}(1/2, 2; d\rho/\rho) \iff \frac{n-1}{\tilde{r}} - \gamma \geq -\frac{1}{r}.$$

Recalling the relation between q, r, p (resp. $\tilde{q}, \tilde{r}, \tilde{p}$) the last condition is equivalent to

$$-\gamma \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} - \frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} \right) - \frac{n}{q} + \frac{n}{p} - n$$

which is precisely the third of conditions (1.7).

The weak Young inequality can be used in (2.9) only in the range $q, r, p \in (1, +\infty)$, which forces

$$1 < p < q < \infty.$$

To cover the cases

$$1 \leq p < q \leq \infty$$

we use instead the strong Young inequality: we can write

$$\| |x|^{-\beta} F * \phi \|_{L^q L^{\bar{q}}} \lesssim \|\phi\|_{L^p L^{\bar{p}}} \left\| \rho^{-\beta + \frac{n}{q}} \|\rho e - \theta\|^{-\gamma} \right\|_{L^{\bar{r}}_{\theta}} \Big\|_{L^r(\rho^{-1} d\rho)} \quad (2.12)$$

for the full range $q, r, p \in [1, +\infty]$. The previous arguments are still valid apart from the last step which must be replaced by

$$\rho^{-\beta + \frac{n}{q}} I_{\gamma \tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq |\rho - 1|^{\frac{n-1}{\tilde{r}} - \gamma} \in L^r(1/2, 2; d\rho/\rho) \iff \frac{n-1}{\tilde{r}} - \gamma > -\frac{1}{r}$$

and this implies that the inequality in the last condition (1.7) must be strict.

The case

$$1 < p = q < \infty$$

has already been covered. Indeed, in this case the scaling condition (1.7) implies

$$\alpha + \beta + \gamma = n \implies \alpha + \beta > 0$$

since $\gamma < n$. Thus when $\tilde{p} = \tilde{q}$ the last inequality in (1.7) is strict and we can apply the second part of the proof; the cases $\tilde{p} \leq \tilde{q}$ follow from the case $\tilde{p} = \tilde{q}$.

To complete the proof, it remains to consider the case (ii) where we assume that the support of the Fourier transform $\hat{\phi}$ is contained in an annular region of size R . By scaling invariance of the inequality, it is sufficient to consider the case $R = 1$. Now let $\psi(x)$ be such that $\hat{\psi} \in C_c^\infty$ and precisely

$$\hat{\psi}(\xi) = 1 \quad \text{for } c'_1 \leq |\xi| \leq c'_2, \quad \hat{\psi}(\xi) = 0 \quad \text{for } |\xi| > 2c'_1 \text{ and } |\xi| < \frac{1}{2}c'_2,$$

for some constants $c'_2 > c_2 \geq c_1 > c'_1 > 0$. This implies

$$\phi = \mathcal{F}^{-1}(\hat{\psi}\hat{\phi}) = \psi * \phi$$

and we can write

$$T_\gamma \phi = |x|^{-\gamma} * \psi * \phi = (T_\gamma \psi) * \phi.$$

Since $T_\gamma \psi = c \mathcal{F}^{-1}(|\xi|^{\gamma-n} \hat{\psi}(\xi))$ is a Schwartz class function, we arrive at the estimates

$$|T_\gamma \phi(x)| \leq C_{\mu, \gamma} \langle x \rangle^{-\mu} * |\phi| \quad \forall \mu \geq 1. \quad (2.13)$$

Here we can take μ arbitrarily large. Thus the proof of case (ii) is concluded by applying the following Lemma:

Lemma 2.3. *Let $n \geq 2$. Assume $1 \leq p \leq q \leq \infty$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$ and α, β, μ satisfy*

$$\beta < \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad \alpha + \beta \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} \right), \quad (2.14)$$

$$\mu > -\alpha - \beta + n \left(1 + \frac{1}{q} - \frac{1}{p} \right). \quad (2.15)$$

Then the following estimate holds:

$$\| |x|^{-\beta} \langle x \rangle^{-\mu} * \phi \|_{L^q_{|x|} L^{\bar{q}}_{\theta}} \lesssim \|\phi\|_{L^p_{|x|} L^{\bar{p}}_{\theta}}. \quad (2.16)$$

Proof. Notice that, by (2.14), the right hand side in (2.15) is always strictly positive and never larger than $n - 1$, thus it is sufficient to prove the lemma for μ in the range

$$0 < \mu \leq n.$$

By (2.8) we have, for all $\tilde{p}, \tilde{q}, \tilde{r} \in [1, +\infty]$ with $1 + 1/\tilde{q} = 1/\tilde{r} + 1/\tilde{p}$,

$$\|\langle \cdot \rangle^{-\mu} * |\phi|(|x|\theta)\|_{L_{\theta}^{\tilde{q}}(\mathbb{S}^{n-1})} \lesssim \int_0^\infty J_{\mu\tilde{r}}(|x|, \rho)^{\frac{1}{\tilde{r}}} \|\phi(\rho\theta)\|_{L_{\theta}^{\tilde{p}}(\mathbb{S}^{n-1})} \rho^{n-1} d\rho. \quad (2.17)$$

Notice that when $\tilde{r} = \infty$ we have

$$\|\langle |x|e - \rho\theta \rangle^{-\mu}\|_{L_{\theta}^{\infty}} \lesssim \langle |x| - \rho \rangle^{-\mu}.$$

We write for brevity

$$Q(|x|) \equiv |x|^{-\beta + \frac{n-1}{q}} \|\langle \cdot \rangle^{-\mu} * |\phi|(|x|\theta)\|_{L_{\theta}^{\tilde{q}}}, \quad P(\rho) = \rho^{\alpha + \frac{n-1}{p}} \|\phi(\rho\theta)\|_{L_{\theta}^{\tilde{p}}}$$

$$J(|x|, \rho) = J_{\mu\tilde{r}}^{\frac{1}{\tilde{r}}}(|x|, \rho) \quad (\text{resp. } \langle |x| - \rho \rangle^{-\mu} \text{ if } \tilde{r} = \infty).$$

Thus (2.17) becomes

$$Q(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \int_0^\infty J(\sigma, \rho) \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho \quad (2.18)$$

and the estimate to be proved (2.16) can be written as

$$\|Q\|_{L^q(0, +\infty)} \lesssim \|P\|_{L^p(0, +\infty)} \quad (2.19)$$

Recall that the integrals of the form $J(\sigma, \rho)$ have been estimated in Lemma 2.2.

We split Q into the sum of several terms corresponding to different regions of ρ, σ . In the region $\sigma \leq 1$ we have $J(\sigma, \rho) \lesssim \langle \rho \rangle^{-\mu}$ so that

$$Q_1(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \int_0^\infty \langle \rho \rangle^{-\mu} \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho \quad (2.20)$$

Thus we see that in this region (2.19) follows simply from Hölder's inequality and the fact that $\alpha < n/p'$ and $\beta < n/q$. Similarly, it is easy to handle the part of the integral with $\rho \leq 1$ since we have then $J(\sigma, \rho) \lesssim \langle \sigma \rangle^{-\mu}$. Thus in the following we can restrict to $\sigma \gtrsim 1, \rho \gtrsim 1$.

When $1 \lesssim \sigma \leq \rho/2$ we have again $J(\sigma, \rho) \lesssim \langle \rho \rangle^{-\mu}$ and (2.18) becomes

$$Q_2(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \int_\sigma^\infty \langle \rho \rangle^{-\mu} \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho \quad (2.21)$$

If we assume

$$\mu > \frac{n}{p'} - \alpha \quad (2.22)$$

we can apply Hölder's inequality and we get

$$Q_3(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{\frac{n}{p'} - \mu - \alpha} \|P\|_{L^p}.$$

Now the right hand side is in $L^q(\sigma \geq 1)$ provided

$$\mu > \frac{n}{p'} - \alpha + \frac{n}{q} - \beta \equiv -\alpha - \beta + n \left(1 + \frac{1}{q} - \frac{1}{p}\right) \quad (2.23)$$

and we see that (2.23) implies (2.22) since $\beta < n/q$ by assumption.

When $1 \lesssim \rho \leq \sigma/2$ we have $J(\sigma, \rho) \lesssim \langle \sigma \rangle^{-\mu}$ and (2.18) becomes

$$Q_4(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{-\mu} \int_0^\sigma \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho \quad (2.24)$$

and by Hölder's inequality we have as before

$$\lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{\frac{n}{p'} - \mu - \alpha} \|P\|_{L^p}$$

so that (2.23) is again sufficient to obtain (2.19).

Finally, let $\sigma \gtrsim 1$, $\rho \gtrsim 1$ and $2^{-1}\sigma \leq \rho \leq 2\sigma$. In this region we must treat differently the values of $\mu\tilde{r}$ larger or smaller than $n-1$, and the case $\tilde{r} = \infty$ is considered at the end. Assume that $n-1 < \mu\tilde{r} \leq n$; then $J(\sigma, \rho) \lesssim \langle \rho \rangle^{1-n} \langle \sigma - \rho \rangle^{\frac{n-1}{\tilde{r}} - \mu}$, and using the relations

$$\sigma \simeq \rho, \quad \frac{1}{\tilde{r}} = 1 + \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}}$$

we see that (2.18) reduces to

$$Q_5(\sigma) \lesssim \sigma^{-\alpha-\beta+(n-1)(\frac{1}{q}-\frac{1}{p}+\frac{1}{\tilde{p}}-\frac{1}{\tilde{q}})} \int_{\sigma/2}^{2\sigma} \langle \sigma - \rho \rangle^{\frac{n-1}{\tilde{r}} - \mu} P(\rho) d\rho. \quad (2.25)$$

The last integral is (bounded by) a convolution of $P(\rho)$ with the function $\langle \rho \rangle^{\frac{n-1}{\tilde{r}} - \mu}$. In order to estimate the $L^q(\sigma \geq 1)$ norm of Q_5 , we use first Hölder's then Young's inequality:

$$\|Q_5\|_{L^q} \lesssim \|\langle \sigma \rangle^{-\epsilon}\|_{L^{q_0}} \|\langle \rho \rangle^{\frac{n-1}{\tilde{r}} - \mu}\|_{L^{q_1}} \|P\|_{L^p}$$

where

$$\epsilon = -\alpha - \beta + (n-1) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} \right), \quad \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{p} - 1.$$

By assumption we have $\epsilon \geq 0$. When $\epsilon > 0$, in order for the norms to be finite we need

$$\epsilon q_0 > 1, \quad \frac{n-1}{\tilde{r}} - \mu < -\frac{1}{q_1}$$

which can be rewritten

$$(n-1) \left(1 + \frac{1}{\tilde{q}} - \frac{1}{\tilde{p}} \right) - \mu + 1 + \frac{1}{q} - \frac{1}{p} < \frac{1}{q_0} < \epsilon$$

and we see that we can find a suitable q_0 provided the first side is strictly smaller than the last side; this condition is precisely equivalent to (2.23) again (recall also that $n-1 < \mu \leq n$). The argument works also in the case $\epsilon = 0$ by choosing $q_0 = \infty$.

If on the other hand $0 < \mu < n-1$, we have $J_{\mu\tilde{r}}^{\frac{1}{\tilde{r}}} \lesssim \langle \rho \rangle^{-\mu}$ also in this region, so that

$$Q_5(\sigma) \lesssim \sigma^{-\beta+\frac{n-1}{q}} \sigma^{\frac{n-1}{p'} - \alpha - \mu} \int_{\sigma/2}^{\sigma} P(\rho) d\rho$$

by $\sigma \simeq \rho$. Hölder's inequality gives

$$Q_5(\sigma) \lesssim \sigma^{-\beta+\frac{n-1}{q}} \sigma^{\frac{n-1}{p'} - \alpha - \mu} \sigma^{\frac{1}{p}} \|P\|_{L^p}$$

which leads to exactly the same computations as above and in the end to (2.23). The case $\mu = n-1$ introduces a logarithmic term which does not change the integrability properties used here.

It remains the last region when $\tilde{r} = \infty$ so that $J(\sigma, \rho) = \langle \sigma - \rho \rangle^{-\mu}$ and $1/\tilde{p} - 1/\tilde{q} = 1$. Then

$$Q_5(\sigma) \lesssim \sigma^{-\beta+\frac{n-1}{q}} \int_{\sigma/2}^{2\sigma} \langle \sigma - \rho \rangle^{-\mu} \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho$$

which is identical with (2.25) with $\tilde{r} = \infty$, thus the same computations apply and the proof is concluded. \square

3. PROOF OF THEOREM 1.8

The Caffarelli-Kohn-Nirenberg inequality is a simple corollary of Theorem (1.3). We begin by taking $0 < a \leq 1$, and indices $r, \tilde{r}, s, \tilde{s}, q, \tilde{q} \in [1, +\infty]$ such that

$$\frac{1}{r} = \frac{a}{s} + \frac{1-a}{q}, \quad \frac{1}{\tilde{r}} = \frac{a}{\tilde{s}} + \frac{1-a}{\tilde{q}}. \quad (3.1)$$

Then by two applications of Hölder's inequality we obtain the interpolation inequality

$$\begin{aligned} \| |x|^{-\gamma} u \|_{L^r L^{\tilde{r}}} &= \| (|x|^{-\delta} u)^a (|x|^{-\beta} u)^{1-a} \|_{L^r L^{\tilde{r}}} \\ &\leq \| (|x|^{-\delta} u)^a \|_{L^{s/a} L^{\tilde{s}/a}} \| (|x|^{-\beta} u)^{1-a} \|_{L^{q/(1-a)} L^{\tilde{q}/(1-a)}} \\ &= \| |x|^{-\delta} u \|_{L^s L^{\tilde{s}}}^a \| |x|^{-\beta} u \|_{L^q L^{\tilde{q}}}^{1-a} \end{aligned} \quad (3.2)$$

provided the exponents γ, δ, β are related by

$$\gamma = a\delta + (1-a)\beta. \quad (3.3)$$

Now the main step of the proof. By Theorem (1.3) we know that

$$\| |x|^{-\delta} T_\lambda u \|_{L^s L^{\tilde{s}}} \lesssim \| |x|^{-\alpha} u \|_{L^p L^{\tilde{p}}}$$

under suitable conditions on the indices. Now using the well known estimate

$$|u(x)| \leq C_{\lambda, n} T_\lambda (|D|^{n-\lambda} u) \quad (3.4)$$

the previous inequality can be equivalently written

$$\| |x|^{-\delta} u \|_{L^s L^{\tilde{s}}} \lesssim \| |x|^{-\alpha} |D|^\sigma u \|_{L^p L^{\tilde{p}}}, \quad \sigma = n - \lambda$$

which together with (3.2) gives

$$\| |x|^{-\gamma} u \|_{L^r L^{\tilde{r}}} \lesssim \| |x|^{-\alpha} |D|^\sigma u \|_{L^p L^{\tilde{p}}}^a \| |x|^{-\beta} u \|_{L^q L^{\tilde{q}}}^{1-a}. \quad (3.5)$$

The conditions on the indices are those given by (3.1), (3.3), plus those listed in the statement of Theorem (1.3) (notice that we are using $-\alpha$ instead of α). The complete list is the following:

$$r, s, q, \tilde{r}, \tilde{s}, \tilde{q} \in [1, +\infty], \quad a < 0 \leq 1, \quad 0 < \sigma < n, \quad (3.6)$$

$$\frac{1}{r} = \frac{a}{s} + \frac{1-a}{q}, \quad \frac{1}{\tilde{r}} = \frac{a}{\tilde{s}} + \frac{1-a}{\tilde{q}}. \quad (3.7)$$

$$1 < s \leq p < \infty, \quad 1 \leq \tilde{s} \leq \tilde{p} \leq \infty, \quad (3.8)$$

$$\gamma < \frac{n}{r}, \quad \beta < \frac{n}{q}, \quad -\alpha < \frac{n}{p}, \quad \delta < \frac{n}{s}, \quad (3.9)$$

$$\gamma = a\delta + (1-a)\beta, \quad (3.10)$$

$$-\alpha + \delta + n - \sigma = n + \frac{n}{s} - \frac{n}{p}, \quad (3.11)$$

$$-\alpha + \delta \geq (n-1) \left(\frac{1}{s} - \frac{1}{p} + \frac{1}{\tilde{p}} - \frac{1}{\tilde{s}} \right). \quad (3.12)$$

Recall also that, when the last inequality (3.12) is strict, we can allow the full range

$$1 \leq s \leq p \leq \infty.$$

Our final task is to rewrite this set of conditions in a compact form, eliminating the redundant parameters δ, s, \tilde{s} . Define the two quantities

$$\Delta = a\sigma + n \left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p} \right), \quad \tilde{\Delta} = a\sigma + n \left(\frac{1}{\tilde{r}} - \frac{1-a}{\tilde{q}} - \frac{a}{\tilde{p}} \right).$$

Then (3.7) are equivalent to

$$\Delta = a \left(\sigma + \frac{n}{s} - \frac{n}{p} \right), \quad \tilde{\Delta} = a \left(\sigma + \frac{n}{\tilde{s}} - \frac{n}{\tilde{p}} \right) \quad (3.13)$$

while (3.11) is equivalent to

$$\delta = \alpha + \frac{\Delta}{a} \quad (3.14)$$

and we can use (3.13), (3.14) to replace δ, s, \tilde{s} in the remaining relations. Condition (3.10) becomes

$$\Delta = \gamma - a\alpha - (1-a)\beta, \quad (3.15)$$

which is precisely the scaling condition, while (3.12) becomes

$$\Delta + (n-1)\tilde{\Delta} \geq 0. \quad (3.16)$$

The last inequality in (3.9), $\delta < n/s$, can be written

$$\alpha < \frac{n}{p} - \sigma$$

so that (3.9) is replaced by

$$\gamma < \frac{n}{r}, \quad \beta < \frac{n}{q}, \quad \frac{n}{p} - n < \alpha < \frac{n}{p} - \sigma. \quad (3.17)$$

Finally, conditions (3.8) translate to

$$1 < p, \quad a \left(\sigma - \frac{n}{p} \right) < \Delta \leq a\sigma, \quad a \left(\sigma - \frac{n}{\tilde{p}} \right) \leq \tilde{\Delta} \leq a\sigma. \quad (3.18)$$

When the inequality in (3.16) is strict, the last condition can be relaxed to

$$1 \leq p, \quad a \left(\sigma - \frac{n}{p} \right) \leq \Delta \leq a\sigma, \quad a \left(\sigma - \frac{n}{\tilde{p}} \right) \leq \tilde{\Delta} \leq a\sigma. \quad (3.19)$$

We pass now to the proof of Corollary 1.9. Assume now σ is integer, and the inequality

$$\| |x|^{-\gamma} u \|_{L^r L^{\tilde{r}}} \leq C \| |x|^{-\alpha} |D|^\sigma u \|_{L^p L^{\tilde{p}}}^a \| |x|^{-\beta} u \|_{L^q L^{\tilde{q}}}^{1-a}$$

is true for a certain choice of the parameters as in the theorem, so that in particular

$$\alpha < \frac{n}{p} - \sigma < \frac{n}{p}.$$

Then we shall prove that also the following inequalities are true

$$\| |x|^{k-\gamma} u \|_{L^r L^{\tilde{r}}} \leq C \| |x|^{k-\alpha} D^\sigma u \|_{L^p L^{\tilde{p}}}^a \| |x|^{k-\beta} u \|_{L^q L^{\tilde{q}}}^{1-a} \quad (3.20)$$

for all integers $k \geq 0$, where we are using the shorthand notation

$$\| |x|^{k-\alpha} D^\sigma u \|_{L^p L^{\tilde{p}}} = \sum_{|\nu|=\sigma} \| |x|^{k-\alpha} D^\nu u \|_{L^p L^{\tilde{p}}}, \quad (\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n).$$

This in particular implies that the condition on α from below can be dropped when σ is integer.

When $k = 0$, (3.20) is obtained just by replacing $|D|^\sigma$ with D^σ in the original inequality. The proof of this estimate is identical to the previous one; the only modification is to use, instead of (3.4), the stronger pointwise bound

$$|u(x)| \leq C_{\lambda,n} T_\lambda (|D^{n-\lambda} u|) \quad (3.21)$$

which is valid for all $\lambda = 1, \dots, n-1$.

Now if we apply (3.20) (with $k = 0$) to a function of the form $|x|^k u$ for some $k \geq 1$, we obtain

$$\| |x|^{k-\gamma} u \|_{L^r L^{\tilde{r}}} \leq C \| |x|^{-\alpha} D^\sigma (|x|^k u) \|_{L^p L^{\tilde{p}}}^a \| |x|^{k-\beta} u \|_{L^q L^{\tilde{q}}}^{1-a}$$

and to conclude the proof we see that it is sufficient to prove the inequality

$$\| |x|^{-\alpha} D^\sigma (|x|^k u) \|_{L^p L^{\tilde{p}}} \lesssim \| |x|^{k-\alpha} D^\sigma u \|_{L^p L^{\tilde{p}}} \quad (3.22)$$

for all $\alpha < n/p$, $1 \leq p, \tilde{p} < \infty$, and integers $\sigma = 1, \dots, n-1$, $k \geq 1$. Notice indeed that all the conditions on the parameters (apart from $\alpha > -n+n/p$) are unchanged if we decrease γ, α, β by the same quantity.

By induction on k (and writing $\delta = -\alpha$), we are reduced to prove that for all $p, \tilde{p} \in [1, \infty)$ and $1 \leq \sigma \leq n-1$

$$\| |x|^\delta D^\sigma(|x|u) \|_{L^p L^{\tilde{p}}} \lesssim \| |x|^{1+\delta} D^\sigma u \|_{L^p L^{\tilde{p}}}, \quad \delta > \sigma - \frac{n}{p}. \quad (3.23)$$

Using Leibnitz' rule we reduce further to

$$\| |x|^{1+\delta-\ell} u \|_{L^p L^{\tilde{p}}} \lesssim \| |x|^{1+\delta} D^\ell u \|_{L^p L^{\tilde{p}}}, \quad \delta > \ell - \frac{n}{p} \quad (3.24)$$

for $\ell = 1, \dots, n-1$, and by induction on ℓ this is implied by

$$\| |x|^\delta u \|_{L^p L^{\tilde{p}}} \lesssim \| |x|^{1+\delta} \nabla u \|_{L^p L^{\tilde{p}}}, \quad \delta > 1 - \frac{n}{p}. \quad (3.25)$$

In order to prove (3.25), consider first the radial case. When $u = \phi(|x|)$ is a radial (smooth compactly supported) function, we have

$$\| |x|^\delta u \|_{L^p L^{\tilde{p}}}^p \simeq \int_0^\infty \rho^{\delta p + n - 1} |\phi(\rho)|^p d\rho.$$

Integrating by parts we get

$$\begin{aligned} &= -\frac{p}{\delta p + n} \int_0^\infty \rho^{\delta p + n} |\phi|^{p-1} |\phi(\rho)|' d\rho \\ &\lesssim \int_0^\infty (\rho^{\delta p + n - 1} |\phi|^p)^{\frac{p-1}{p}} (\rho^{\delta p + p + n - 1} |\phi'|^p)^{\frac{1}{p}} d\rho \\ &\simeq \| |x|^\delta u \|_{L^p L^{\tilde{p}}}^{\frac{p-1}{p}} \| |x|^{1+\delta} \nabla u \|_{L^p L^{\tilde{p}}} \end{aligned}$$

which implies (3.25) in the radial case. If u is not radial, define

$$\phi(\rho) = \| u(\rho\theta) \|_{L_{\theta}^{\tilde{p}}(\mathbb{S}^{n-1})} = \left(\int_{\mathbb{S}^{n-1}} |u(\rho\theta)|^{\tilde{p}} dS_{\theta} \right)^{\frac{1}{\tilde{p}}}$$

so that

$$\| |x|^\delta u \|_{L^p L^{\tilde{p}}} \simeq \left(\int_0^\infty \rho^{\delta p + n - 1} |\phi(\rho)|^p d\rho \right)^{\frac{1}{p}}.$$

The proof in the radial case implies

$$\| |x|^\delta u \|_{L^p L^{\tilde{p}}} \leq \| |x|^{\delta+1} \phi'(|x|) \|_{L^p};$$

moreover we have

$$\begin{aligned} |\phi'(\rho)| &\lesssim \phi^{1-\tilde{p}} \int_{\mathbb{S}^{n-1}} |u(\rho\theta)|^{\tilde{p}-1} |\theta \cdot \nabla u| dS_{\theta} \\ &\leq \phi^{1-\tilde{p}} \left(\int_{\mathbb{S}} |u|^{\tilde{p}} \right)^{\frac{\tilde{p}-1}{\tilde{p}}} \left(\int_{\mathbb{S}} |\nabla u|^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} = \| \nabla u(\rho\theta) \|_{L_{\theta}^{\tilde{p}}(\mathbb{S}^{n-1})} \end{aligned}$$

and in conclusion we obtain

$$\| |x|^\delta u \|_{L^p L^{\tilde{p}}} \leq \| |x|^{\delta+1} \nabla u \|_{L^p L^{\tilde{p}}}$$

as claimed.

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